International Journal of Applied Mathematics
& Statistical Sciences (IJAMSS)
ISSN(P): 2319-3972; ISSN(E): 2319-3980
Vol. 2, Issue 5, Nov 2013, 93-116

International Academy of Science, Engineering and Technology
Connecting Researchers; Nurturing Innovations

UNSTEADY MHD MIXED CONVECTION OF A VISCOUS DOUBLE DIFFUSIVE FLUID OVER A VERTICAL PLATE IN POROUS MEDIUM WITH CHEMICAL REACTION, THERMAL RADIATION AND JOULE HEATING

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ABSTRACT

The problem of 2-dimensional unsteady boundary layer MHD mixed double diffusive flow of a viscous incompressible, electrically conducting fluid along a semi-infinite vertical permeable moving plate in presence of a transverse magnetic field, chemical reaction, heat absorption and thermal radiation is considered. The dimensionless governing partial differential equations for this study are solved analytically by using 2-term harmonic and non-harmonic functions. Furthermore, the plate is assumed to move with a constant velocity in fluid flow direction while the free stream velocity is assumed to follow the perturbation rule. Numerical evaluation of the analytical results are performed and few graphical results for the velocity, temperature and concentration distributions and tabulated results for Skin friction, Nusselt number and Sherwood numbers are discussed and presented.

KEYWORDS: MHD, Mixed Convection, Heat and Mass Transfer, Thermal Radiation, Heat Absorption, Thermal Diffusivity, Permeability and Chemical Reaction

1. INTRODUCTION

Mixed convection flows with simultaneous heat and mass transfer in porous media under the influence of a magnetic field and chemical reaction are frequently encountered in many transport processes in nature. Its application is found in many industries viz. in the chemical industry, power and cooling industry for drying, chemical vapour deposition on surfaces, cooling of nuclear reactors and magneto hydrodynamic power generators. Simultaneous heat transfer and evaporation of crude oil in different stages of refining process are physical examples of heat and mass transfer. Many transport processes exist in nature and in industrial applications in which the simultaneous heat and mass transfer occurs as a result of combined buoyancy effects of diffusion of chemical species. We are particularly interested in cases in which diffusion and chemical reaction occur at roughly the same speed. When diffusion is much faster than chemical reaction, then only chemical factors influence the chemical reaction rate. When diffusion is not much faster than reaction, the diffusion and kinetics interact to produce very different effects. The study of heat generation or absorption effects in moving fluids is important in view of several physical problems, such as fluids undergoing exothermic or endothermic chemical reaction. Thermal diffusion effect has been utilized for isotopes separation in the mixture between gases with very light molecular weight (hydrogen and helium) and medium molecular weight. Chambre and Young [1] have presented a first order chemical reaction in the neighborhood of a horizontal plate. Dekha et al. [2] investigated the effect of the first order homogeneous chemical reaction on the process of an unsteady flow past a vertical plate with a constant heat and mass transfer. Muthucumaraswamy [3] presented heat and mass transfer effects on a continuously moving isothermal vertical surface with uniform suction by taking into account the homogeneous chemical reaction of first order. A comprehensive description of the theoretical work for both laminar and turbulent mixed convection boundary layer flows has been given in a review paper by Chen and Armaly [4] and in the book by Pop and Ingham [5]. The problem of mixed convection under the influence of magnetic field has attracted numerous researchers viz. Soundalgekar et al. [6], Elbasheshy [7], Abel et al. [8, 9] in view of its applications in geophysics and astrophysics. The effect of radiation on MHD flow and heat transfer problems has become industrially more important. At high operating temperatures, radiation effect can be quite significant. The effect of variable viscosity on hydro magnetic flow and heat transfer past a continuously moving porous boundary with radiation has been studied by Seddeek [10].

The same author investigated [11] thermal radiation and buoyancy effects on MHD free convective heat generating flow over an accelerating permeable surface with temperature-dependent. The radiation effects on MHD free -convection flow of a gas past a semi-infinite vertical plate is studied by Takhar et al. [12]. The fundamental problem of flow through and past porous media has been discussed by Cheng [13] and Rudraiah [14] on thermal radiation as a mode of energy transfer and emphasize the need for inclusion of radiactive transfer in these processes.

In most chemical reactions, the reaction rate depends on the concentration of the species itself. A chemical reaction is said to be of first order, if the rate of reaction is directly proportional to concentration itself. During chemical reaction between two species concentration heat is also generated. Chemical reaction effects on heat and mass transfer laminar boundary layer flow have been discussed by various authors [15, 16, 17, 18, and 19] in various situations. Ramachandra Prasad and Bhaskar Reddy [20] investigated radiation and mass transfer effects on unsteady MHD free convection flow past a heated vertical plate in a porous medium with viscous dissipation. Seddeek et al. [21] analyzed the effects of chemical reaction, radiation and variable viscosity on hydromagnetic mixed convection heat and mass transfer for Hiemenz flow through porous media. Ibrahim et al. [22] analyzed the effects of the chemical reaction and radiation absorption on the unsteady MHD free convection flow past a semi-infinite vertical permeable moving plate with heat source and suction Chaudhary et al. [23] have analyzed the effect of radiation on heat transfer in MHD mixed convection flow with simultaneous thermal and mass diffusion from an infinite vertical plate with viscous dissipation. Recently, Pal and Talukdar [24] studied the combined effect of MHD and Ohmic heating in unsteady two-dimensional boundary layer slip flow, heat and mass transfer of a viscous incompressible fluid past a vertical permeable plate with the diffusion of species in the presence of thermal radiation incorporating the first-order chemical reaction. Madhusudhana Rao B, Viswanatha reddy G and Raju M.C [25] studied MHD transient free convection and chemically reactive flow past a porous vertical plate with radiation and temperature gradient dependent heat source in slip flow regime.

2. MATHEMATICAL ANALYSIS

We consider a two-dimensional unsteady free convection flow of an incompressible viscous fluid past an infinite vertical porous plate. In rectangular Cartesian coordinate system, we take x-axis along the plate in the direction of flow and y-axis normal to it. It is assumed that the plate moves with a constant velocity in the flow direction in the presence of a transverse applied magnetic field. It is also assumed that the temperature and the concentration at the wall as well as the suction velocity are exponentially varying with time. The equations of continuity, linear momentum, angular momentum, energy and diffusion, which govern the flow field, are solved by using a regular perturbation method.

The behavior of the velocity, temperature, concentration, skin-friction, Nusselt number and Sherwood number has been discussed for variations in the physical parameters. Further the flow is considered in presence of temperature gradient dependent heat source and effect of radiation and chemical reaction.

Introduce the boundary layer and Boussineq's approximations. Under the above assumptions, the equations governing the conservation of mass (continuity), momentum, energy and concentration can be taken as follows.

$$\frac{\partial v^1}{\partial y^1} = 0 \tag{1}$$

$$\frac{\partial u^1}{\partial t^1} + v^1 \frac{\partial u^1}{\partial y^1} = -\frac{1}{\rho}\frac{\partial p^1}{\partial x^1} + g\beta_T (T^1 - T^1_\infty) + g\beta_c (C^1 - C^1_\infty) + \upsilon \frac{\partial^2 u^1}{\partial y^{1^2}} - \left(\frac{\upsilon}{k^1} + \frac{\sigma B_0{}^2}{\rho}\right) u^1 \tag{2}$$

$$\frac{\partial T^1}{\partial t^1} + v^1 \frac{\partial T^1}{\partial y^1} = \alpha \frac{\partial^2 T^1}{\partial y^{1^2}} + \left(\frac{\upsilon}{C_p k^1} + \frac{\sigma B_0{}^2}{\rho C_p}\right) u^{1^2} + \frac{\upsilon}{C_p}\left(\frac{\partial u^1}{\partial y^1}\right)^2 - \frac{1}{\rho C_p}\frac{\partial q_r}{\partial y^1} - \frac{Q_0}{\rho C_p}(T^1 - T^1_\infty) \tag{3}$$

$$\frac{\partial C^1}{\partial t^1} + v^1 \frac{\partial C^1}{\partial y^1} = D \frac{\partial^2 C^1}{\partial y^{1^2}} - R(C^1 - C^1_\infty) + D_1 \frac{\partial^2 T^1}{\partial y^{1^2}} \tag{4}$$

The boundary conditions relevant to the problem are;

$u^1 = U^1_p$, $T^1 = T^1_w + \varepsilon(T^1_w - T^1_w)e^{n^1t^1}$, $C^1 = C^1_w + \varepsilon(C^1_w - C^1_w)e^{n^1t^1}$ at $y^1 = 0$ $u^1 \to U^1_\infty = U_0(1 + \varepsilon e^{n^1t^1})$,

$T^1 \to T^1_\infty$, $C^1 \to C^1_\infty$ as $y^1 \to \infty$ (5)

Where u^1 and v^1 are the components of velocity along x-axis and y-axis directions, t is the time, g is the acceleration due to gravity, β_T and β_c are the coefficients of thermal expansion and concentration expansion respectively, υ is the kinematic viscosity, k^1 is the permeability of the porous medium, ρ is the density of the fluid, σ is the magnetic permeability of the fluid , B_0 is the uniform magnetic field, T^1 is dimensional temperature, α is the fluid thermal diffusivity, C_p is the specific heat at constant pressure, q_r is the radioactive heat flux, Q_0 is the dimensional heat source, T^1_w is the temperature of the wall as well as the temperature of the fluid at the plate, T^1_∞ is the temperature of the fluid far away from the plate, C^1 is dimensional concentration, D is the molecular diffusivity, C^1_∞ is the concentration of the wall as well as the concentration of the fluid at the plate. U^1_p is the wall dimensional velocity, U^1_∞ is the free stream dimensional velocity, U_0 and n^1 are constants, R is chemical reaction parameter and D_1 is thermal diffusivity

The equation of continuity (1) yields that the suction velocity v^1 at the plate is either a constant or function of time; hence it takes exponential form;

$$v^1 = -V_0(1 + \varepsilon A e^{n^1t^1}) \tag{6}$$

Where V_0>0 is a scale of suction velocity at the plate and n^1 is a positive constant, here the negative sign indicates that the suction velocity acts towards the plate, A is a real positive constant, ε and εA are very small quantities less than unity.

Outside of the boundary layer, the equation (2) gives that

$$-\frac{1}{\rho}\frac{\partial p^1}{\partial x^1}=\frac{dU_\infty^1}{dt^1}+\frac{\upsilon}{k^1}U_\infty^1+\frac{\sigma B_0{}^2}{\rho}U_\infty^1 \tag{7}$$

The fourth and fifth terms on RHS of energy equation (3) denote the thermal radiation effect and heat absorption effects respectively, The radio active heat flux is given by A.C. Cogley [25] as

$$\frac{\partial q_r}{\partial y^1}=4(T^1-T_\infty^1)I^1 \tag{8}$$

Where $I^1=\int_0^\infty K_\lambda \frac{\partial e_\lambda}{\partial T^1}d\lambda$, K_λ is the absorption coefficient at the wall and e_λ is plank's function.

3. METHOD OF SOLUTION

The following non-dimensional variables are employed,

$$u=\frac{u^1}{U_0},\ v=\frac{v^1}{V_0}\quad y=\frac{y^1V_0}{\upsilon},\ U_\infty=\frac{U_\infty^1}{U_0},\ U_p=\frac{U_p^1}{U_0},\ K=\frac{V_0{}^2K^1}{\upsilon^2}\quad t=\frac{V_0{}^2t^1}{\upsilon}$$

$$n=\frac{\upsilon n^1}{V_0{}^2},\ T=\frac{T^1-T_\infty^1}{T_w^1-T_\infty^1},\ C=\frac{C^1-C_\infty^1}{C_w^1-C_\infty^1}, \tag{9}$$

In presence of the above discussed equations (6)--(9), the general equations (2)--(4) take the form;

$$\frac{\partial u}{\partial t}-(1+\varepsilon Ae^{nt})\frac{\partial u}{\partial y}=\frac{dU_\infty}{dt}+\frac{\partial^2 u}{\partial y^2}+N(U_\infty-u)+G_TT+G_cC \tag{10}$$

$$\frac{\partial T}{\partial t}-(1+\varepsilon Ae^{nt})\frac{\partial T}{\partial y}=\frac{1}{P_r}\frac{\partial^2 T}{\partial y^2}-(S+Q)T+NE_cu^2+E_c\left(\frac{\partial u}{\partial y}\right)^2 \tag{11}$$

$$\frac{\partial C}{\partial t}-(1+\varepsilon Ae^{nt})\frac{\partial C}{\partial y}=\frac{1}{S_c}\frac{\partial^2 C}{\partial y^2}+S_0\frac{\partial^2 T}{\partial y^2}-\gamma C \tag{12}$$

Where, $G_T=\dfrac{g\upsilon\beta_T(T_w^1-T_\infty^1)}{U_0V_0{}^2}$, $G_c=\dfrac{g\upsilon\beta_c(C_w^1-C_\infty^1)}{U_0V_0{}^2}$, $Q=\dfrac{\upsilon Q_0}{\rho C_pV_0{}^2}$, $S=\dfrac{4I^1\upsilon}{\rho C_pV_0{}^2}$,

$P_r=\dfrac{\upsilon}{\alpha}=\dfrac{\upsilon C_p\rho}{K}$, $N=M+\dfrac{1}{K}$, $M=\dfrac{\upsilon B_0{}^2\sigma}{\rho V_0{}^2}$, $E_c=\dfrac{U_0{}^2}{C_p\left(T_w^1-T_\infty^1\right)}$, $S_c=\dfrac{\upsilon}{D}$, $S_0=\dfrac{D_1}{\upsilon}\left(\dfrac{T_w^1-T_\infty^1}{C_w^1-C_\infty^1}\right)$,

$\gamma=\dfrac{\upsilon R}{V_0{}^2}$

Here G_T is thermal Grashof number, G_c is the solutal Grashof number, Q is dimensionless heat absorption coefficient, S is the radiation parameter, Pr is Prandtl number, M is magnetic field parameter, K is a permeability parameter of the porous medium, E_c is Eckert number, Sc is the Schmidit number, S_0 is Soret effect parameter and γ is Chemical reaction parameter. Moreover, the dimensionless grammar of the boundary conditions (5) takes the form;

$$u = U_p,\ T = 1 + \varepsilon e^{nt},\ C = 1 + \varepsilon e^{nt},\ \text{at}\ y^1 = 0$$

$$u \to U_\infty = 1 + \varepsilon e^{nt},\ T \to 0,\ C \to 0,\ \text{as}\ y^1 \to \infty \tag{13}$$

4. SOLUTION OF THE PROBLEM

The partial differential equations (10), (11), (12) cannot be solved in closed form. So we solve analytically by converting them into ordinary differential equations in dimension less grammar. Therefore the expressions for velocity, temperature and concentration can be represented in the following form.

$$u(y,t) = f_0(y) + \varepsilon e^{nt} f_1(y) + O(\varepsilon^2) \tag{14}$$

$$T(y,t) = g_0(y) + \varepsilon e^{nt} g_1(y) + O(\varepsilon^2) \tag{15}$$

$$C(y,t) = h_0(y) + \varepsilon e^{nt} h_1(y) + O(\varepsilon^2) \tag{16}$$

Substituting above equations (14)--(16) into (10)--(12), equating the harmonic and non-harmonic terms that is coefficients of $\varepsilon^0, \varepsilon^1$ and neglecting the higher order terms of $O(\varepsilon^2)$, one obtains the following pairs of ordinary differential equations for (f_0,g_0,h_0) and (f_1,g_1,h_1).

$$f_0^{11} + f_0^1 - Nf_0 = -N - G_T g_0 - G_c h_0 \tag{17}$$

$$f_1^{11} + f_1^1 - (N+n) f_1 = -Af_0^1 - G_T g_1 - G_c h_1 - (N+n) \tag{18}$$

$$g_0^{11} + P_r g_0^1 - P_r (S+Q) g_0 = -P_r E_c \left(N f_0^{\ 2} + f_0^{1^2} \right) \tag{19}$$

$$g_1^{11} + P_r g_1^1 - P_r (S+Q+n) g_1 = -AP_r g_0^1 - 2P_r E_c \left(N f_0 f_1 + f_0^1 f_1^1 \right) \tag{20}$$

$$\frac{1}{S_c} h_0^{11} + h_0^1 - \gamma h_0 = -S_0 g_0^{11} \tag{21}$$

$$\frac{1}{S_c} h_1^{11} + h_1^1 - (n+\gamma) h_1 = -Ah_0^1 - S_0 g_1^{11} \tag{22}$$

With the boundary conditions,

$$f_0 = U_p,\ f_1 = 0,\ g_0 = 1,\ g_1 = 1,\ h_0 = 1,\ h_1 = 1 \quad at\ \ y = 0$$

$$f_0 \to 1,\ f_1 \to 1,\ g_0 \to 0,\ g_1 \to 0,\ h_0 \to 0,\ h_1 \to 0 \quad as\ \ y \to \infty \tag{23}$$

But the above equations (17)--(22) are still coupled. Therefore, we expand the flow variables as an asymptotic series solution about the Eckert number E_c (E_c<<1 for an incompressible flow)

$$f_0(y) = f_{00}(y) + E_c f_{01}(y) + O\left(E_c^{\ 2}\right)$$

$$f_1(y) = f_{10}(y) + E_c f_{11}(y) + O\left(E_c^{\,2}\right)$$

$$g_0(y) = g_{00}(y) + E_c g_{01}(y) + O\left(E_c^{\,2}\right) \tag{24}$$

$$g_1(y) = g_{10}(y) + E_c g_{11}(y) + O\left(E_c^{\,2}\right)$$

$$h_0(y) = h_{00}(y) + E_c h_{01}(y) + O\left(E_c^{\,2}\right)$$

$$h_1(y) = h_{10}(y) + E_c h_{11}(y) + O\left(E_c^{\,2}\right)$$

On substituting above equations (24) in to (17)--(22), equating the coefficients of $E_c^{\,0}$, $E_c^{\,1}$ and neglecting higher order terms of E_c, we obtain the following sequence of ordinary differential equations.

Zero Order

$$f_{00}^{\,11} + f_{00}^{\,1} - Nf_{00} = -N - G_T g_{00} - G_c h_{00} \tag{25}$$

$$f_{10}^{\,11} + f_{10}^{\,1} - (N+n) f_{10} = -(N+n) - Af_{00}^{1} - G_T g_{10} - G_c h_{10} \tag{26}$$

$$g_{00}^{\,11} + P_r g_{00}^{\,1} - P_r (S+Q) g_{00} = 0 \tag{27}$$

$$g_{10}^{\,11} + P_r g_{10}^{\,1} - P_r (S+Q+n) g_{10} = -AP_r g_{00}^{\,1} \tag{28}$$

$$\frac{1}{S_c} h_{00}^{\,11} + h_{00}^{\,1} - \gamma h_{00} = -S_0 g_{00}^{\,11} \tag{29}$$

$$\frac{1}{S_c} h_{10}^{\,11} + h_{10}^{\,1} - (n+\gamma) h_{10} = -Ah_{00}^{1} - S_0 g_{10}^{\,11} \tag{30}$$

First Order

$$f_{01}^{\,11} + f_{01}^{\,1} - Nf_{01} = -G_T g_{01} - G_c h_{01} \tag{31}$$

$$f_{11}^{\,11} + f_{11}^{\,1} - (N+n) f_{11} = -Af_{01}^{\,1} - G_T g_{11} - G_c h_{11} \tag{32}$$

$$g_{01}^{\,11} + P_r g_{01}^{\,1} - P_r (S+Q) g_{01} = -P_r (Nf_{00}^{\,2} + f_{00}^{1\,2}) \tag{33}$$

$$g_{11}^{\,11} + P_r g_{11}^{\,1} - P_r (S+Q+n) g_{11} = -AP_r g_{01}^{1} - 2P_r \left(Nf_{00} f_{10} + f_{00}^{\,1} f_{10}^{\,1}\right) \tag{34}$$

$$\frac{1}{S_c} h_{01}^{\,11} + h_{01}^{\,1} - \gamma h_{01} = S_0 g_{01}^{\,11} \tag{35}$$

$$\frac{1}{S_c} h_{11}^{\,11} + h_{11}^{\,1} - (n+\gamma) h_{11} = -Ah_{01}^{1} - S_0 g_{11}^{\,11} \tag{36}$$

And the boundary conditions (23) take the shape,

$$f_{00} = U_p,\ f_{10} = f_{01} = f_{11} = 0$$
$$g_{00} = g_{10} = 1, g_{01} = g_{11} = 0$$
$$h_{00} = h_{10} = 1, h_{01} = h_{11} = 0 \qquad \text{At } y=0$$

$$f_{00} \to f_{10} \to 1,\ f_{01} \to f_{11} \to 0$$
$$g_{00} \to g_{10} \to g_{01} \to g_{11} \to 0 \qquad \text{As } y \to \infty$$
$$h_{00} \to h_{10} \to h_{01} \to h_{11} \to 0 \tag{37}$$

On solving the 2nd order differential equations with constant coefficients (25)--(36) under the boundary conditions (37), we get

$$f_{00} = 1 + A_{33}e^{-m_4 y} + A_3 e^{-m_1 y} + A_4 e^{-m_3 y} \tag{38}$$

$$f_{10} = 1 + A_{39}e^{-m_8 y} + A_{34}e^{-m_4 y} + A_{35}e^{-m_1 y} + A_{36}e^{-m_3 y} + A_{37}e^{-m_2 y} + A_{38}e^{-m_7 y} \tag{39}$$

$$g_{00} = e^{-m_1 y} \tag{40}$$

$$g_{10} = A_{27}e^{-m_2 y} + A_1 e^{-m_1 y} \tag{41}$$

$$h_{00} = A_{28}e^{-m_3 y} + A_2 e^{-m_1 y} \tag{42}$$

$$h_{10} = A_{32}e^{-m_7 y} + A_{29}e^{-m_3 y} + A_{30}e^{-m_2 y} + A_{31}e^{-m_1 y} \tag{43}$$

$$\begin{aligned} f_{01} &= A_{51}e^{-m_9 y} + A_{40} + A_{41}e^{-m_5 y} + A_{42}e^{-2m_4 y} + A_{43}e^{-2m_1 y} + A_{44}e^{-2m_3 y} \\ &+ A_{45}e^{-m_1 y} + A_{46}e^{-(m_1+m_3)y} + A_{47}e^{-m_3 y} + A_{48}e^{-m_4 y} + A_{49}e^{-(m_1+m_4)y} + A_{50}e^{-(m_3+m_4)y} \end{aligned} \tag{44}$$

$$\begin{aligned} f_{11} &= A_{126}e^{-m_{12} y} + A_{100}e^{-m_9 y} + A_{101}e^{-m_5 y} + A_{102}e^{-2m_4 y} + A_{103}e^{-2m_1 y} + A_{104}e^{-2m_3 y} + A_{105}e^{-m_1 y} \\ &+ A_{106}e^{-(m_3+m_1)y} + A_{107}e^{-m_3 y} + A_{108}e^{-m_4 y} + A_{109}e^{-(m_1+m_4)y} + A_{110}e^{-(m_3+m_4)y} + A_{111}e^{-m_{10} y} + A_{112}e^{-(m_4+m_8)y} \\ &+ A_{113}e^{-(m_2+m_4)y} + A_{114}e^{-(m_7+m_4)y} + A_{115}e^{-m_8 y} + A_{116}e^{-m_2 y} + A_{117}e^{-m_7 y} + A_{118}e^{-(m_1+m_8)y} + A_{119}e^{-(m_1+m_2)y} \\ &+ A_{120}e^{-(m_1+m_7)y} + A_{121}e^{-(m_3+m_8)y} + A_{122}e^{-(m_2+m_3)y} + A_{123}e^{-(m_3+m_7)y} + A_{124}e^{-m_{11} y} + A_{125}e^{-m_6 y} \end{aligned} \tag{45}$$

$$\begin{aligned} g_{01} &= A_{15}e^{-m_5 y} + A_5 + A_6 e^{-2m_4 y} + A_7 e^{-2m_1 y} + A_8 e^{-2m_3 y} + A_9 e^{-m_1 y} \\ &+ A_{10}e^{-(m_1+m_3)y} + A_{11}e^{-m_3 y} + A_{12}e^{-m_4 y} + A_{13}e^{-(m_1+m_4)y} + A_{14}e^{-(m_3+m_4)y} \end{aligned} \tag{46}$$

$$\begin{aligned} g_{11} &= A_{74}e^{-m_{10} y} + A_{52}e^{-m_5 y} + A_{53}e^{-2m_4 y} + A_{54}e^{-2m_1 y} + A_{55}e^{-2m_3 y} + A_{56}e^{-m_1 y} + A_{57}e^{-(m_3+m_1)y} \\ &+ A_{58}e^{-m_3 y} + A_{59}e^{-m_4 y} + A_{60}e^{-(m_1+m_4)y} + A_{61}e^{-(m_3+m_4)y} + A_{62}e^{-(m_4+m_8)y} + A_{63}e^{-(m_2+m_4)y} \\ &+ A_{64}e^{-(m_7+m_4)y} + A_{65}e^{-m_8 y} + A_{66}e^{-m_2 y} + A_{67}e^{-m_7 y} + A_{68}e^{-(m_1+m_8)y} + A_{69}e^{-(m_1+m_2)y} \\ &+ A_{70}e^{-(m_1+m_7)y} + A_{71}e^{-(m_3+m_8)y} + A_{72}e^{-(m_2+m_3)y} + A_{73}e^{-(m_3+m_7)y} \end{aligned} \tag{47}$$

$$\begin{aligned}h_{01} &= A_{26}e^{-m_6y} + A_{16}e^{-m_5y} + A_{17}e^{-2m_4y} + A_{18}e^{-2m_1y} + A_{19}e^{-2m_3y}\\ &+ A_{20}e^{-m_1y} + A_{21}e^{-(m_1+m_3)y} + A_{22}e^{-m_3y} + A_{23}e^{-m_4y} + A_{24}e^{-(m_1+m_4)y} + A_{25}e^{-(m_3+m_4)y}\end{aligned} \tag{48}$$

$$\begin{aligned}h_{11} &= A_{99}e^{-m_{11}y} + A_{75}e^{-m_6y} + A_{76}e^{-m_5y} + A_{77}e^{-2m_4y} + A_{78}e^{-2m_1y} + A_{79}e^{-2m_3y} + A_{80}e^{-m_1y} + A_{81}e^{-(m_3+m_1)y}\\ &+ A_{82}e^{-m_3y} + A_{83}e^{-m_4y} + A_{84}e^{-(m_1+m_4)y} + A_{85}e^{-(m_3+m_4)y} + A_{86}e^{-m_{10}y} + A_{87}e^{-(m_4+m_8)y} + A_{88}e^{-(m_2+m_4)y}\\ &+ A_{89}e^{-(m_7+m_4)y} + A_{90}e^{-m_8y} + A_{91}e^{-m_2y} + A_{92}e^{-m_7y} + A_{93}e^{-(m_1+m_8)y} + A_{94}e^{-(m_1+m_2)y} + A_{95}e^{-(m_1+m_7)y}\\ &+ A_{96}e^{-(m_3+m_8)y} + A_{97}e^{-(m_2+m_3)y} + A_{98}e^{-(m_3+m_7)y}\end{aligned} \tag{49}$$

Where the constants are given in the appendix. In view of the above solutions (38)--(49) and the equations (24) & (14)--(16), the velocity distribution u (y, t), the temperature distribution T(y, t) and the concentration distribution C(y, t) in the boundary layer are given as,

$$\begin{aligned}u(y,t) &= \left(1 + A_{33}e^{-m_4y} + A_3e^{-m_1y} + A_4e^{-m_3y}\right) + E_c\begin{pmatrix} A_{51}e^{-m_9y} + A_{40} + A_{41}e^{-m_5y} + A_{42}e^{-2m_4y} + A_{43}e^{-2m_1y} + A_{44}e^{-2m_3y} + A_{45}e^{-m_1y}\\ + A_{46}e^{-(m_1+m_3)y} + A_{47}e^{-m_3y} + A_{48}e^{-m_4y} + A_{49}e^{-(m_1+m_4)y} + A_{50}e^{-(m_3+m_4)y}\end{pmatrix}\\ &+ \varepsilon e^{nt}\begin{bmatrix} \left(1 + A_{39}e^{-m_8y} + A_{34}e^{-m_4y} + A_{35}e^{-m_1y} + A_{36}e^{-m_3y} + A_{37}e^{-m_2y} + A_{38}e^{-m_7y}\right)\\ + E_c\begin{pmatrix} A_{126}e^{-m_{12}y} + A_{100}e^{-m_9y} + A_{101}e^{-m_5y} + A_{102}e^{-2m_4y} + A_{103}e^{-2m_1y} + A_{104}e^{-2m_3y} + A_{105}e^{-m_1y}\\ + A_{106}e^{-(m_3+m_1)y} + A_{107}e^{-m_3y} + A_{108}e^{-m_4y} + A_{109}e^{-(m_1+m_4)y} + A_{110}e^{-(m_3+m_4)y} + A_{111}e^{-m_{10}y} + A_{112}e^{-(m_4+m_8)y}\\ + A_{113}e^{-(m_2+m_4)y} + A_{114}e^{-(m_7+m_4)y} + A_{115}e^{-m_8y} + A_{116}e^{-m_2y} + A_{117}e^{-m_7y} + A_{118}e^{-(m_1+m_8)y} + A_{119}e^{-(m_1+m_2)y}\\ + A_{120}e^{-(m_1+m_7)y} + A_{121}e^{-(m_3+m_8)y} + A_{122}e^{-(m_2+m_3)y} + A_{123}e^{-(m_3+m_7)y} + A_{124}e^{-m_{11}y} + A_{125}e^{-m_6y}\end{pmatrix}\end{bmatrix}\end{aligned} \tag{50}$$

$$\begin{aligned}T(y,t) &= \left(e^{-m_1y}\right) + E_c\begin{pmatrix} A_{15}e^{-m_5y} + A_5 + A_6e^{-2m_4y} + A_7e^{-2m_1y} + A_8e^{-2m_3y} + A_9e^{-m_1y}\\ + A_{10}e^{-(m_1+m_3)y} + A_{11}e^{-m_3y} + A_{12}e^{-m_4y} + A_{13}e^{-(m_1+m_4)y} + A_{14}e^{-(m_3+m_4)y}\end{pmatrix}\\ &+ \varepsilon e^{nt}\begin{bmatrix} \left(A_{27}e^{-m_2y} + A_1e^{-m_1y}\right) + E_c\begin{pmatrix} A_{74}e^{-m_{10}y} + A_{52}e^{-m_5y} + A_{53}e^{-2m_4y} + A_{54}e^{-2m_1y} + A_{55}e^{-2m_3y} + A_{56}e^{-m_1y} + A_{57}e^{-(m_3+m_1)y}\\ + A_{58}e^{-m_3y} + A_{59}e^{-m_4y} + A_{60}e^{-(m_1+m_4)y} + A_{61}e^{-(m_3+m_4)y} + A_{62}e^{-(m_4+m_8)y} + A_{63}e^{-(m_2+m_4)y}\\ + A_{64}e^{-(m_7+m_4)y} + A_{65}e^{-m_8y} + A_{66}e^{-m_2y} + A_{67}e^{-m_7y} + A_{68}e^{-(m_1+m_8)y} + A_{69}e^{-(m_1+m_2)y}\\ + A_{70}e^{-(m_1+m_7)y} + A_{71}e^{-(m_3+m_8)y} + A_{72}e^{-(m_2+m_3)y} + A_{73}e^{-(m_3+m_7)y}\end{pmatrix}\end{bmatrix}\end{aligned} \tag{51}$$

$$\begin{aligned}C(y,t) &= \left(A_{28}e^{-m_3y} + A_2e^{-m_1y}\right) + E_c\begin{pmatrix} A_{26}e^{-m_6y} + A_{16}e^{-m_5y} + A_{17}e^{-2m_4y} + A_{18}e^{-2m_1y} + A_{19}e^{-2m_3y} + A_{20}e^{-m_1y}\\ + A_{21}e^{-(m_1+m_3)y} + A_{22}e^{-m_3y} + A_{23}e^{-m_4y} + A_{24}e^{-(m_1+m_4)y} + A_{25}e^{-(m_3+m_4)y}\end{pmatrix}\\ &+ \varepsilon e^{nt}\begin{bmatrix} \left(A_{32}e^{-m_7y} + A_{29}e^{-m_3y} + A_{30}e^{-m_2y} + A_{31}e^{-m_1y}\right)\\ + E_c\begin{pmatrix} A_{99}e^{-m_{11}y} + A_{75}e^{-m_6y} + A_{76}e^{-m_5y} + A_{77}e^{-2m_4y} + A_{78}e^{-2m_1y} + A_{79}e^{-2m_3y} + A_{80}e^{-m_1y} + A_{81}e^{-(m_3+m_1)y}\\ + A_{82}e^{-m_3y} + A_{83}e^{-m_4y} + A_{84}e^{-(m_1+m_4)y} + A_{85}e^{-(m_3+m_4)y} + A_{86}e^{-m_{10}y} + A_{87}e^{-(m_4+m_8)y} + A_{88}e^{-(m_2+m_4)y}\\ + A_{89}e^{-(m_7+m_4)y} + A_{90}e^{-m_8y} + A_{91}e^{-m_2y} + A_{92}e^{-m_7y} + A_{93}e^{-(m_1+m_8)y} + A_{94}e^{-(m_1+m_2)y} + A_{95}e^{-(m_1+m_7)y}\\ + A_{96}e^{-(m_3+m_8)y} + A_{97}e^{-(m_2+m_3)y} + A_{98}e^{-(m_3+m_7)y}\end{pmatrix}\end{bmatrix}\end{aligned} \tag{52}$$

SKIN FRICTION

The expression for the rate of mass transfer in terms of skin-friction (τ) at the plate is,

$$\tau = \left(\frac{du}{dy}\right)_{y=0} = \left(\frac{df_0}{dy}\right)_{y=0} + \varepsilon e^{nt}\left(\frac{df_1}{dy}\right)_{y=0}$$

$$= A_{127} + \varepsilon e^{nt} A_{128} \tag{53}$$

NUSSELT NUMBER

The expression for the rate of heat transfer in terms of Nusselt number (N_u) is,

$$N_u = \left(\frac{dT}{dy}\right)_{y=0} = \left(\frac{dg_0}{dy}\right)_{y=0} + \varepsilon e^{nt}\left(\frac{dg_1}{dy}\right)_{y=0}$$

$$= A_{129} + \varepsilon e^{nt} A_{130} \tag{54}$$

SHERWOOD NUMBER

The expression for the stretching sheet (S_h) is,

$$S_h = \left(\frac{dC}{dy}\right)_{y=0} = \left(\frac{dh_0}{dy}\right)_{y=0} + \varepsilon e^{nt}\left(\frac{dh_1}{dy}\right)_{y=0}$$

$$= A_{131} + \varepsilon e^{nt} A_{132} \tag{55}$$

5. RESULTS AND DISCUSSIONS

The non-linear equations (14) to (22) subject to the boundary conditions (23) describing heat and mass transfer double diffusive flow past a semi infinite vertical plate immersed in a porous medium in the presence of thermal radiation, chemical reaction and joul heating under the influence of magnetic field are solved analytically by perturbation method. In order to find physical insight of the problem, the effects of various parameters (Magnetic parameter M, Heat absorption parameter Q, Chemical reaction parameter γ, Schimidt number S_c, Radiation parameter S, Prandtl number Pr, Thermal Grashof number G_T, Solutal Grashof number G_c, Permeability number K and Soret number S_0) are analyzed on Velocity, Temperature and Concentration distributions with the help of following figures and tables.

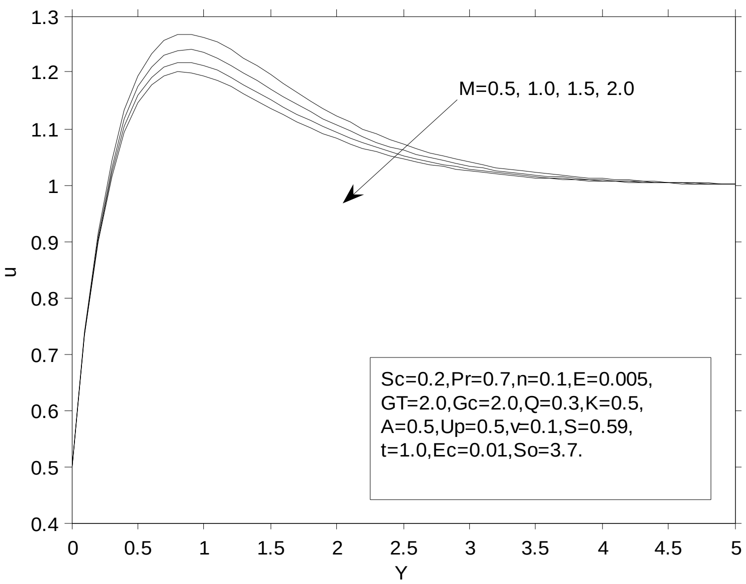

Figure 1: Effect of Magnetic Field M on Velocity u

From Figure 1, it is observed that the velocity decreases as the existence of magnetic field (M) increases, because the application of transverse magnetic field results in Lorentz force. In Figure 2 the velocity decreases as the heat absorption (Q) increases and this causes the thermal buoyancy effects to decrease, which results in a total reduction in the fluid velocity. In the figures 3&4, it is seen that the velocity decreases as the chemical reaction (γ) and Schimidt number (S_c) increase respectively. The effects of radiation parameter (S) and Prandtl number (Pr) on velocity distribution are presented in the figures 5&6. From these figures it is noticed that as S and Pr increase, the velocity decreases due to decrease in boundary layer thickness for t=1.0. The velocity distribution for distinct values of Thermal Grashof number (G_T) and Solutal Grashof number (G_c) are shown in the figures 7&8 respectively for t=1.0, and it seen that an increase in G_T and G_c lead to increase the velocity profile. From these figures it is concluded that the peak values of velocity increases rapidly near the porous plate's wall as the Grashof number is increased which ultimately decays to the relevant free stream velocity. From the Figure 9, it is observed that due to increase in the plate velocity (Up) there is an increase in the velocity near the porous plate and its effects diminish away from the plate. The resistance offered by the porous medium decreases as the permeability of porous medium increase, and so it is observed that as Permeability parameter (K) increases, the velocity increases across the boundary layer from the Figure 10.

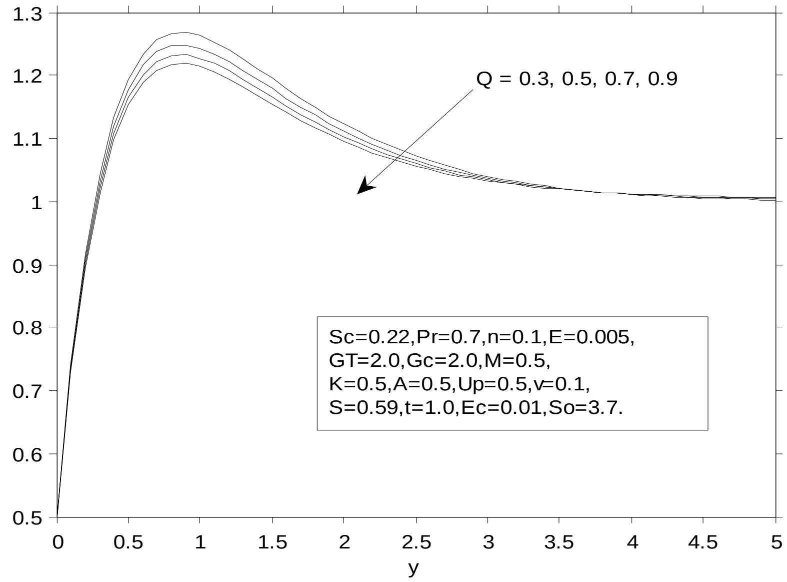

Figure 2: Effect of Heat Absorption Parameter Q on Velocity u

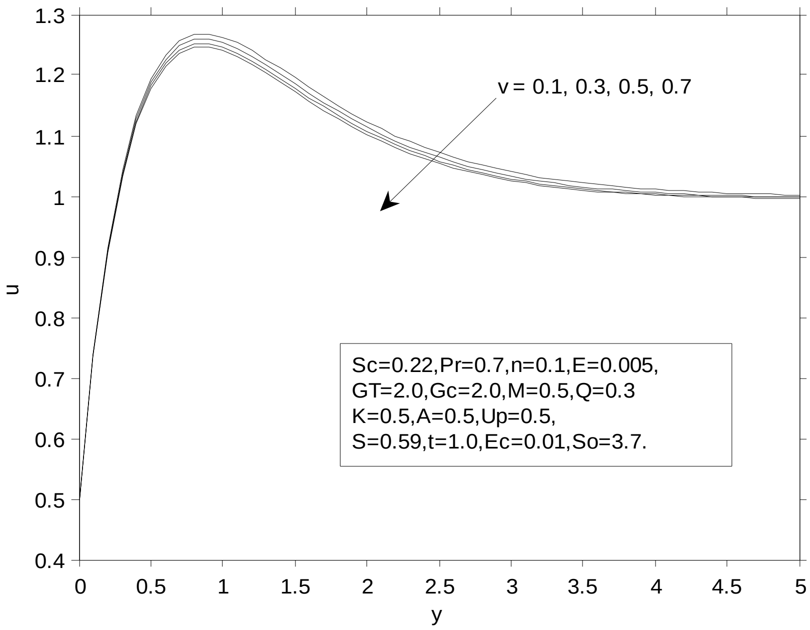

Figure 3: Effect of Chemical Reaction Parameter γ and on Velocity u

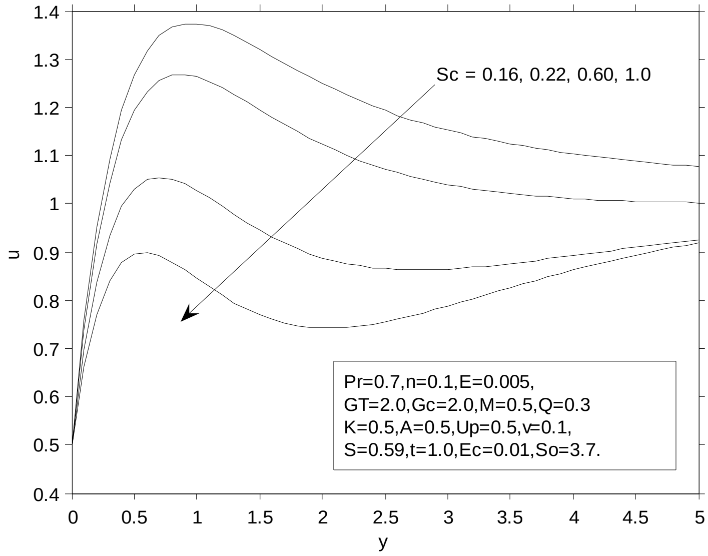

Figure 4: Effect of Schimidt Number S_c on Velocity u

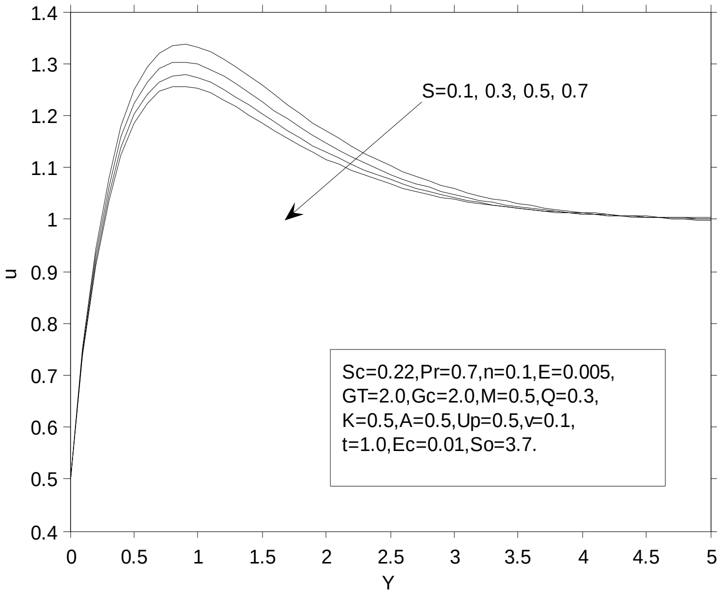

Figure 5: Effect of Radiation Parameter S on Velocity u

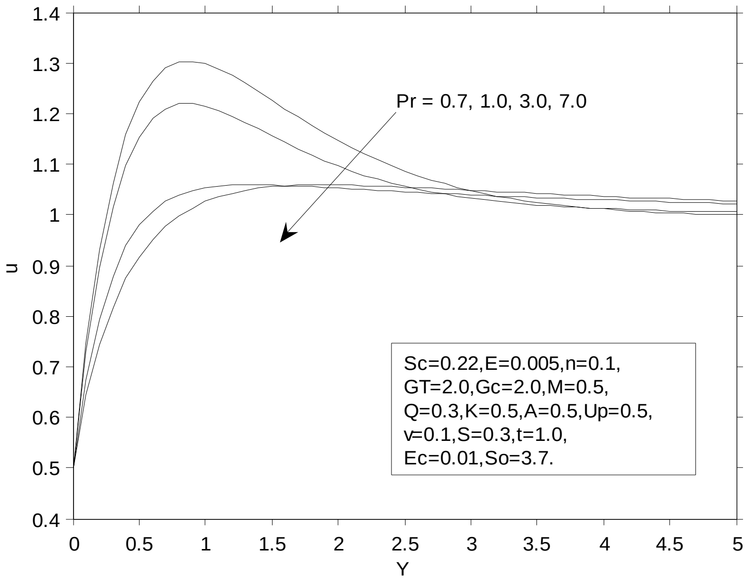

Figure 6: Effect of and Prandtl Number Pr on Velocity u

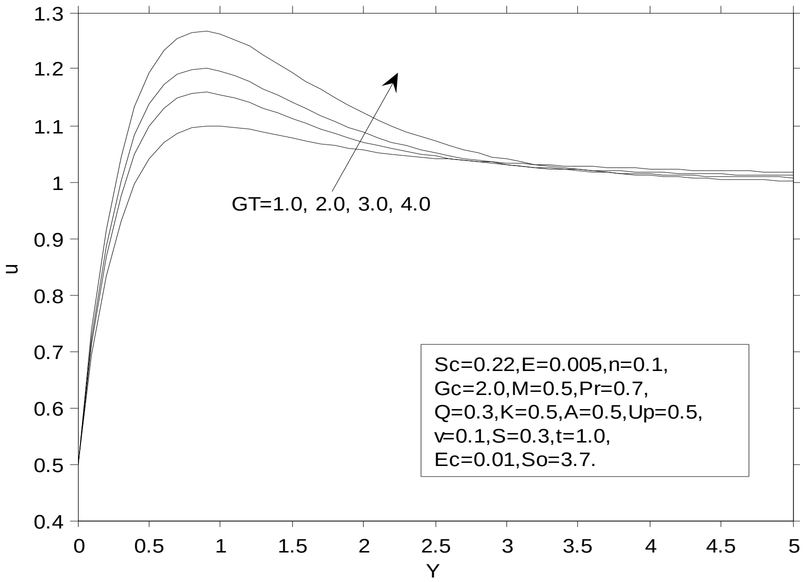

Figure 7: Effect of Thermal Grashof Number G_T on Velocity u

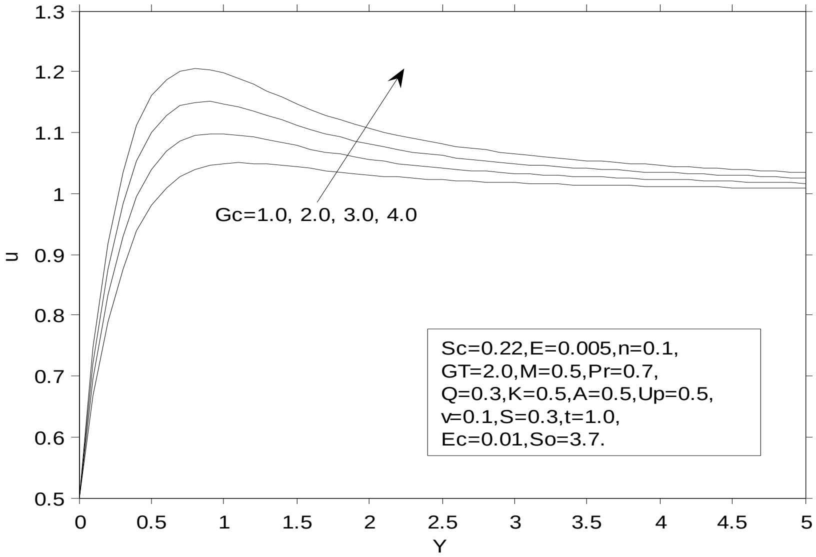

Figure 8: Effect of Solutal Grashof Number G_c on Velocity u

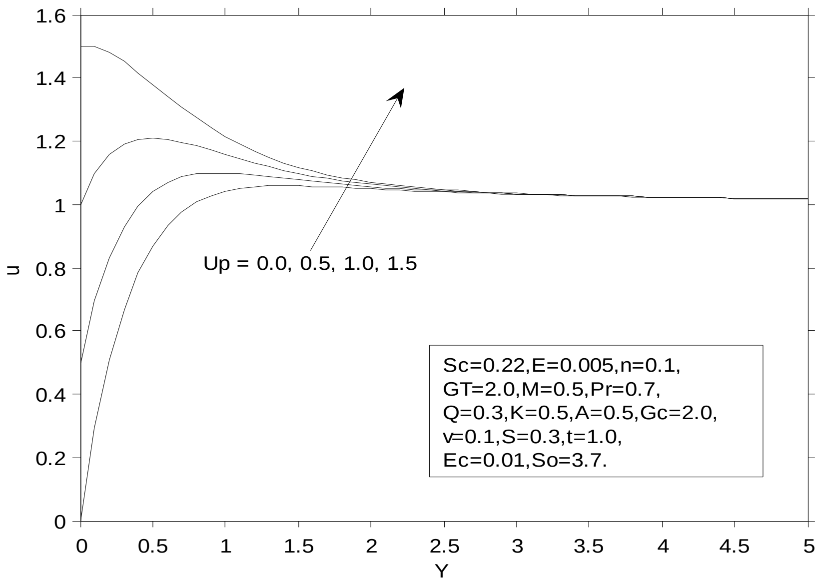

Figure 9: Effect of Plate Velocity U_p on Velocity u

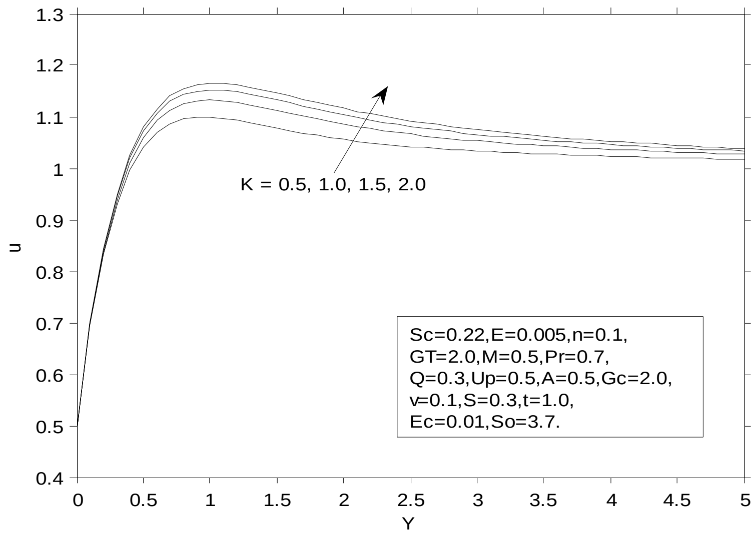

Figure 10: Effect of Permeability Parameter K on Velocity u

From Figure 11, it is observed that temperature profile decreases as heat absorption (Q) increases. This is due to the fact that heat of the fluid is absorbed by the porous plate and hence higher the heat absorption parameter, lower the temperature profile in the boundary layer. In Figure 12, it is seen that the increase in the radiation parameter (S) results in decrease in temperature due to the fact that the divergence of radiation heat flux $\frac{\partial q_r}{\partial y^1}$ decreases as the absorption coefficient K_λ increases at the wall which causes the fluid temperature to decrease. It is noticed from Figure 13 that temperature profile decreases as Prandtl number(Pr) increases, and this is due to fact that thermal boundary layer decreases with an increase in Pr. In figure 14, it is shown that temperature distribution decreases with an increase in an increase in Soret effect number (S_o).

From Figure 15, it is observed that the concentration decreases as the chemical reaction parameter (γ) increases, and this is due to the fact that solutal boundary layer decreases with chemical reaction parameter. In the Figures 16&17, it is seen that concentration profile decreases as Schimidt number (Sc) and Soret number (So) increase respectively.

The variations in Skin friction (τ), Nusselt number (N_u) and Sherwood number (S_h) against respective parameters are discussed and presented in the Tables 1, 2 & 3.

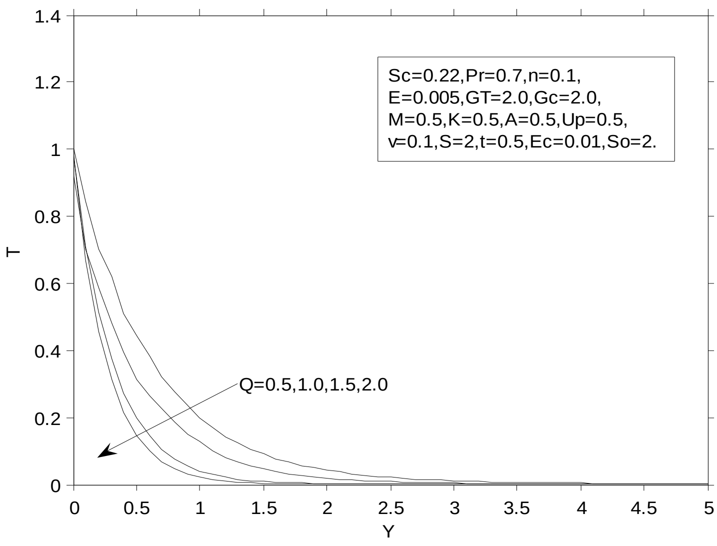

Figure 11: Effect of Heat Absorption Parameter Q on Temperature T

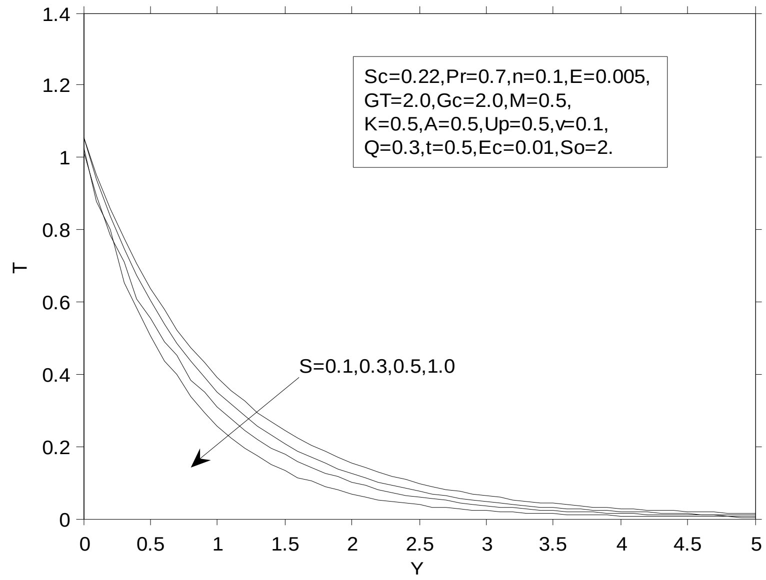

Figure 12: Effect of Radiation Parameter S on Temperature T

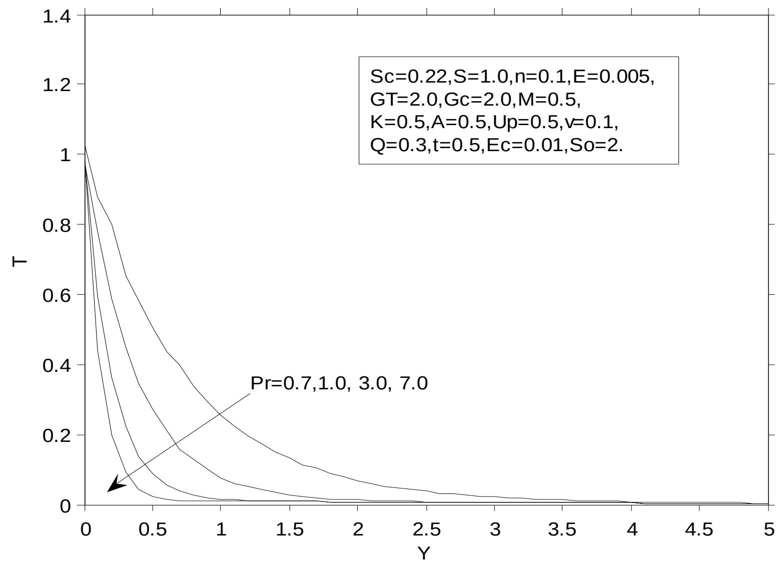

Figure 13: Effect of Prandtl Number Pr on Temperature T

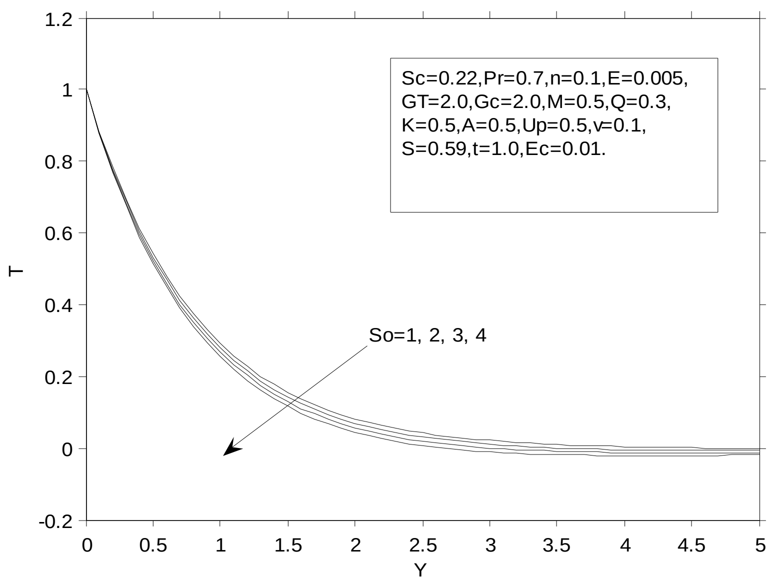

Figure 14: Effect of Soret Number S_o on Temperature T

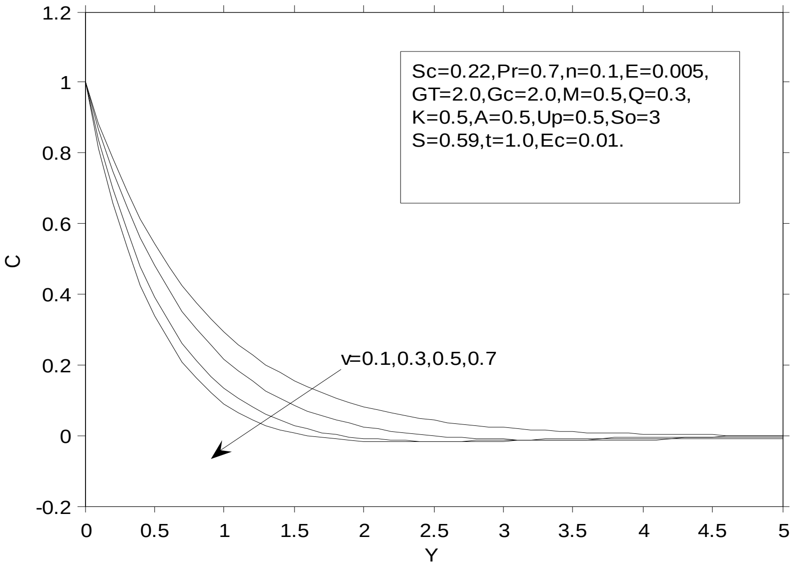

Figure 15: Effect of Chemical Reaction Parameter γ on Concentration C

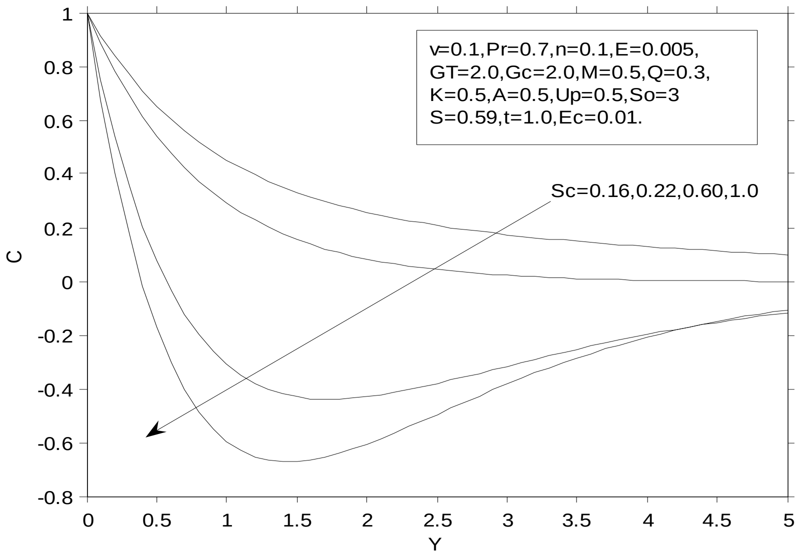

Figure 16: Effect of Schimidt Number S_c on Concentration C

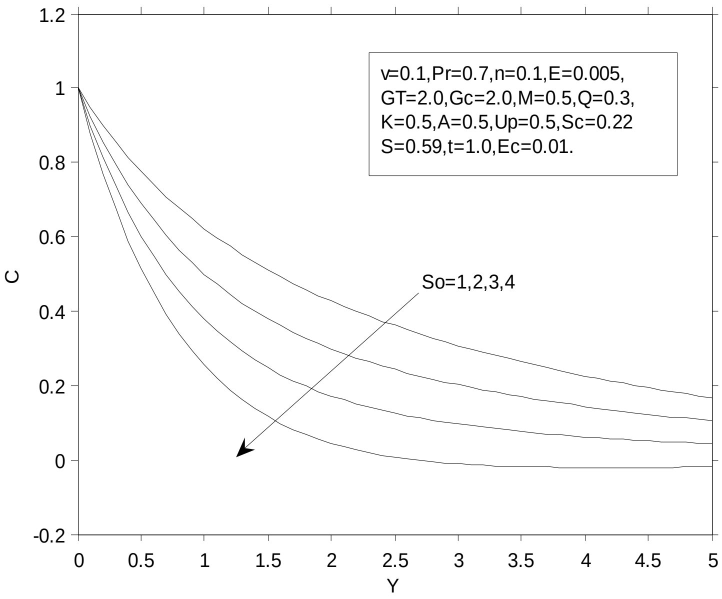

Figure 17: Effect of Soret Number S_o on Concentration C

Table 1

k	S	Q	γ	Sc	Pr	τ
0.5	0.5	0.5	0.1	0.16	0.7	1.9110
1.0	0.5	0.5	0.1	0.16	0.7	2.7631
1.5	0.5	0.5	0.1	0.16	0.7	3.0389
0.5	1.0	0.5	0.1	0.16	0.7	-1.0379
0.5	1.5	0.5	0.1	0.16	0.7	-2.0677
0.5	0.5	1.0	0.1	0.16	0.7	-1.0687
0.5	0.5	1.5	0.1	0.16	0.7	-2.1685
0.5	0.5	0.5	0.2	0.16	0.7	-2.1231
0.5	0.5	0.5	0.3	0.16	0.7	-2.3945
0.5	0.5	0.5	0.1	0.22	0.7	-3.3399
0.5	0.5	0.5	0.1	0.60	0.7	-3.7276
0.5	0.5	0.5	0.1	0.16	1.0	-6.9262
0.5	0.5	0.5	0.1	0.16	3.0	-8.0790

Table 2

Q	S	P_r	S_o	N_u
0.5	0.5	0.7	2	-1.1971
1.0	0.5	0.7	2	-1.5138
1.5	0.5	0.7	2	-1.5917
0.5	1.0	0.7	2	-1.5258
0.5	1.5	0.7	2	-1.5827
0.5	0.5	1.0	2	-1.2789
0.5	0.5	3.0	2	-1.4195
0.5	0.5	0.7	3	-1.7849
0.5	0.5	0.7	4	-1.9039

Table 3

γ	Sc	S_o	S_h
0.1	0.16	2	-0.7849
0.2	0.16	2	-0.9977
0.3	0.16	2	-1.1238
0.1	0.22	2	-0.8241
0.1	0.60	2	-0.9148
0.1	0.16	3	-0.9271
0.1	0.16	4	-1.0039

6. CONCLUSIONS

The governing partial differential equations were transformed to ordinary differential equations by suitable transformation and then equations were solved analytically by perturbation method. The computed values obtained from analytical solutions of Velocity, Temperature and Concentration fields and are discussed graphically. The Skin friction, Nusselt number and Sherwood number are presented in tabular form. Thus, we conclude the following.

- The Velocity profile decreases as the values of Magnetic parameter, Heat absorption parameter, Chemical reaction parameter, Schimidt number, Radiation parameter, Prandtl number increase where as it increases with an increase in Permeability number of porous medium, Thermal and Solutal Grashof numbers.
- The Temperature distribution decreases as the values of Heat absorption parameter, Radiation parameter, Prandtl number and Soret effect number increase in the boundary layer.
- The Concentration distribution decreases as the values of Chemical reaction parameter, Schimidt number and Soret effect number increase.

- The value of Skin friction decreases as Radiation parameter, Heat absorption parameter, Chemical reaction parameter, Schimidit number, prandtl number increase and it increases with an increase in permeability number.
- The value of Nusselt number decreases as Radiation parameter, Heat absorption parameter, prandtl number and Soret effect number increases.
- The value of Sherwood number decreases as Chemical reaction parameter, Radiation parameter, Soret effect number increase.

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APPENDICES

$$m_1 = m_5 = \frac{P_r + \sqrt{P_r^{\,2} + 4P_r(S+Q)}}{2} \qquad m_2 = m_{10} = \frac{P_r + \sqrt{P_r^{\,2} + 4P_r(S+Q+n)}}{2}$$

$$m_3 = m_6 = \frac{S_c + \sqrt{S_c^{\,2} + 4\gamma S_c}}{2} \qquad m_4 = m_9 = \frac{1 + \sqrt{1+4N}}{2}$$

$$m_7 = m_{11} = \frac{S_c + \sqrt{S_c^{\,2} + 4(n+\gamma)S_c}}{2} \qquad m_8 = m_{12} = \frac{1 + \sqrt{1+4(n+N)}}{2}$$

$$A_1 = \frac{AP_r m_1}{m_1^2 - P_r m_1 - P_r(S+Q+n)} \qquad A_{27} = 1 - A_1 \qquad A_2 = \frac{S_o m_1^2}{\left(\frac{1}{S_c}\right)m_1^2 - m_1 - \gamma} \qquad A_{28} = 1 - A_2$$

$$A_3 = \frac{-G_T - A_2 G_c}{m_1^2 - m_1 - N} \quad A_4 = \frac{A_2 G_c - G_c}{m_3^2 - m_3 - N} \quad A_{33} = U_p - 1 - A_3 - A_4 \quad A_5 = \frac{N}{S+Q}$$

$$A_6 = \frac{(-N - m_4^2)P_r A_{33}^2}{4m_4^2 - 2P_r m_4 - P_r(S+Q)} \quad A_7 = \frac{(-N - m_1^2)P_r A_3^2}{4m_1^2 - 2P_r m_1 - P_r(S+Q)} \quad A_8 = \frac{(-N - m_3^2)P_r A_4^2}{4m_3^2 - 2P_r m_3 - P_r(S+Q)}$$

$$A_9 = \frac{-2P_r N A_3}{m_1^2 - P_r m_1 - P_r(S+Q)} \quad A_{10} = \frac{(-N - m_1 m_3)2P_r A_3 A_4}{(m_1+m_3)^2 - P_r(m_1+m_3) - P_r(S+Q)}$$

$$A_{11} = \frac{-2P_r N A_4}{m_3^2 - P_r m_3 - P_r(S+Q)} \quad A_{12} = \frac{-2P_r N A_{33}}{m_4^2 - P_r m_4 - P_r(S+Q)}$$

$$A_{13} = \frac{(-N - m_1 m_4)2P_r A_3 A_{33}}{(m_1+m_4)^2 - P_r(m_1+m_4) - P_r(S+Q)} \quad A_{14} = \frac{(-N - m_3 m_4)2P_r A_4 A_{33}}{(m_3+m_4)^2 - P_r(m_3+m_4) - P_r(S+Q)}$$

$$A_{15} = -A_5 - A_6 - A_7 - A_8 - A_9 - A_{10} - A_{11} - A_{12} - A_{13} - A_{14} \quad A_{16} = \frac{S_0 m_5^2 A_{15}}{\left(\frac{1}{S_c}\right)m_5^2 - m_5 - \gamma}$$

$$A_{17} = \frac{4S_0 m_4^2 A_6}{\left(\frac{4}{S_c}\right)m_4^2 - 2m_4 - \gamma} \quad A_{18} = \frac{4S_0 m_1^2 A_7}{\left(\frac{4}{S_c}\right)m_1^2 - 2m_1 - \gamma} \quad A_{19} = \frac{4S_0 m_3^2 A_8}{\left(\frac{4}{S_c}\right)m_3^2 - 2m_3 - \gamma}$$

$$A_{20} = \frac{S_0 m_1^2 A_9}{\left(\frac{1}{S_c}\right)m_1^2 - m_1 - \gamma} \quad A_{21} = \frac{S_0 (m_1+m_3)^2 A_{10}}{\left(\frac{1}{S_c}\right)(m_1+m_3)^2 - (m_1+m_3) - \gamma} \quad A_{22} = \frac{S_0 m_3^2 A_{11}}{\left(\frac{1}{S_c}\right)m_3^2 - m_3 - \gamma}$$

$$A_{23} = \frac{S_0 m_4^2 A_{12}}{\left(\frac{1}{S_c}\right)m_4^2 - m_4 - \gamma} \quad A_{24} = \frac{S_0 (m_1+m_4)^2 A_{13}}{\left(\frac{1}{S_c}\right)(m_1+m_4)^2 - (m_1+m_4) - \gamma}$$

$$A_{25} = \frac{S_0 (m_3+m_4)^2 A_{14}}{\left(\frac{1}{S_c}\right)(m_3+m_4)^2 - (m_3+m_4) - \gamma}$$

$$A_{26} = -A_{16} - A_{17} - A_{18} - A_{19} - A_{20} - A_{21} - A_{22} - A_{23} - A_{24} - A_{25} \quad A_{29} = \frac{AA_{28} m_3}{\left(\frac{1}{S_c}\right)m_3^2 - m_3 - (n+\gamma)}$$

$$A_{30} = \frac{-S_0 A_{27} m_2^{\,2}}{\left(\frac{1}{S_c}\right) m_2^{\,2} - m_2 - (n+\gamma)} \quad A_{31} = \frac{AA_2 m_1 - S_0 A_1 m_1^{\,2}}{\left(\frac{1}{S_c}\right) m_1^{\,2} - m_1 - (n+\gamma)} \quad A_{32} = 1 - A_{29} - A_{30} - A_{31}$$

$$A_{34} = \frac{AA_{33} m_4}{m_4^{\,2} - m_4 - (n+N)} \quad A_{35} = \frac{AA_3 m_1 - G_T A_1 - G_c A_{31}}{m_1^{\,2} - m_1 - (n+N)} \quad A_{36} = \frac{AA_4 m_3 - G_c A_{29}}{m_3^{\,2} - m_3 - (n+N)}$$

$$A_{37} = \frac{-G_T A_{27} - G_c A_{30}}{m_2^{\,2} - m_2 - (n+N)} \quad A_{38} = \frac{-G_c A_{32}}{m_7^{\,2} - m_7 - (n+N)} \quad A_{39} = -1 - A_{34} - A_{35} - A_{36} - A_{37} - A_{38}$$

$$A_{40} = \frac{G_T A_5}{N} \quad A_{41} = \frac{-G_T A_{15} - G_c A_{16}}{m_5^{\,2} - m_5 - N} \quad A_{42} = \frac{-G_T A_6 - G_c A_{17}}{4m_4^{\,2} - 2m_4 - N} \quad A_{43} = \frac{-G_T A_7 - G_c A_{18}}{4m_1^{\,2} - 2m_1 - N}$$

$$A_{44} = \frac{-G_T A_8 - G_c A_{19}}{4m_3^{\,2} - 2m_3 - N} \quad A_{45} = \frac{-G_T A_9 - G_c A_{20}}{m_1^{\,2} - m_1 - N} \quad A_{46} = \frac{-G_T A_{10} - G_c A_{21}}{(m_1 + m_3)^2 - (m_1 + m_3) - N}$$

$$A_{47} = \frac{-G_T A_{11} - G_c A_{22}}{m_3^{\,2} - m_3 - N} \quad A_{48} = \frac{-G_T A_{12} - G_c A_{23}}{m_4^{\,2} - m_4 - N} \quad A_{49} = \frac{-G_T A_{13} - G_c A_{24}}{(m_1 + m_4)^2 - (m_1 + m_4) - N}$$

$$A_{50} = \frac{-G_T A_{14} - G_c A_{25}}{(m_3 + m_4)^2 - (m_3 + m_4) - N}$$

$$A_{51} = -A_{40} - A_{41} - A_{42} - A_{43} - A_{44} - A_{45} - A_{46} - A_{47} - A_{48} - A_{49} - A_{50}$$ \

$$A_{52} = \frac{AP_r A_{15} m_5}{m_5^{\,2} - P_r m_5 - P_r (S + Q + n)} \quad A_{53} = \frac{2P_r (AA_6 m_4 - NA_{33} A_{34} - A_{33} A_{34} m_4^{\,2})}{4m_4^{\,2} - 2P_r m_4 - P_r (S + Q + n)}$$

$$A_{54} = \frac{2P_r (AA_7 m_1 - NA_3 A_{35} - A_3 A_{35} m_1^{\,2})}{4m_1^{\,2} - 2P_r m_1 - P_r (S + Q + n)} \quad A_{55} = \frac{2P_r (AA_8 m_3 - NA_4 A_{36} - A_4 A_{36} m_3^{\,2})}{4m_3^{\,2} - 2P_r m_3 - P_r (S + Q + n)}$$

$$A_{56} = \frac{P_r (AA_9 m_1 - 2NA_{35} - 2NA_3)}{m_1^{\,2} - P_r m_1 - P_r (S + Q + n)}$$

$$A_{57} = \frac{P_r \left[AA_{10}(m_1 + m_3) - 2NA_3 A_{36} - 2NA_4 A_{35} - 2A_3 A_{36} m_1 m_3 - 2A_4 A_{35} m_1 m_3\right]}{(m_1 + m_3)^2 - P_r (m_1 + m_3) - P_r (S + Q + n)}$$

$$A_{58} = \frac{P_r (AA_{11} m_3 - 2NA_{36} - 2NA_4)}{m_3^{\,2} - P_r m_3 - P_r (S + Q + n)} \quad A_{59} = \frac{P_r (AA_{12} m_4 - 2NA_{33} - 2NA_{34})}{m_4^{\,2} - P_r m_4 - P_r (S + Q + n)}$$

$$A_{60} = \frac{P_r \left[AA_{13}(m_1 + m_4) - 2NA_{33} A_{35} - 2NA_3 A_{34} - 2A_{33} A_{35} m_1 m_4 - 2A_3 A_{34} m_1 m_4\right]}{(m_1 + m_4)^2 - P_r (m_1 + m_4) - P_r (S + Q + n)}$$

$$A_{61} = \frac{P_r \left[AA_{14}(m_3 + m_4) - 2NA_{33} A_{36} - 2NA_4 A_{34} - 2A_{33} A_{36} m_3 m_4 - 2A_4 A_{34} m_3 m_4\right]}{(m_3 + m_4)^2 - P_r (m_3 + m_4) - P_r (S + Q + n)}$$

$$A_{62}=\frac{-2P_rA_{33}A_{39}(N+m_8m_4)}{(m_8+m_4)^2-P_r(m_8+m_4)-P_r(S+Q+n)}$$

$$A_{63}=\frac{-2P_rA_{33}A_{37}(N+m_2m_4)}{(m_2+m_4)^2-P_r(m_2+m_4)-P_r(S+Q+n)}$$

$$A_{64}=\frac{-2P_rA_{33}A_{39}(N+m_7m_4)}{(m_7+m_4)^2-P_r(m_7+m_4)-P_r(S+Q+n)}\quad A_{65}=\frac{-2P_rNA_{39}}{m_8^{\ 2}-P_rm_8-P_r(S+Q+n)}$$

$$A_{66}=\frac{-2P_rNA_{37}}{m_2^{\ 2}-P_rm_2-P_r(S+Q+n)}\qquad A_{67}=\frac{-2P_rNA_{38}}{m_7^{\ 2}-P_rm_7-P_r(S+Q+n)}$$

$$A_{68}=\frac{-2P_rA_3A_{39}(N+m_8m_1)}{(m_8+m_1)^2-P_r(m_8+m_1)-P_r(S+Q+n)}$$

$$A_{69}=\frac{-2P_rA_3A_{37}(N+m_2m_1)}{(m_2+m_1)^2-P_r(m_2+m_1)-P_r(S+Q+n)}$$

$$A_{70}=\frac{-2P_rA_3A_{38}(N+m_7m_1)}{(m_7+m_1)^2-P_r(m_7+m_1)-P_r(S+Q+n)}$$

$$A_{71}=\frac{-2P_rA_4A_{39}(N+m_8m_3)}{(m_8+m_3)^2-P_r(m_8+m_3)-P_r(S+Q+n)}\quad A_{72}=\frac{-2P_rA_4A_{37}(N+m_2m_3)}{(m_2+m_3)^2-P_r(m_2+m_3)-P_r(S+Q+n)}$$

$$A_{73}=\frac{-2P_rA_4A_{38}(N+m_7m_3)}{(m_7+m_3)^2-P_r(m_7+m_3)-P_r(S+Q+n)}$$

$$A_{74}=-A_{52}-A_{53}-A_{54}-A_{55}-A_{56}-A_{57}-A_{58}-A_{59}-A_{60}-A_{61}-A_{62}-A_{63}-A_{64}-A_{65}-A_{66}-A_{67}-A_{68}$$
$$-A_{69}-A_{70}-A_{71}-A_{72}-A_{73}$$

$$A_{75}=\frac{AA_{26}m_6}{\left(\frac{1}{S_c}\right)m_6^{\ 2}-m_6-(n+\gamma)}\quad A_{76}=\frac{AA_{16}m_5-S_0A_{52}m_5^{\ 2}}{\left(\frac{1}{S_c}\right)m_5^{\ 2}-m_5-(n+\gamma)}\quad A_{77}=\frac{2AA_{17}m_4-4S_0A_{53}m_4^{\ 2}}{\left(\frac{4}{S_c}\right)m_4^{\ 2}-2m_4-(n+\gamma)}$$

$$A_{78}=\frac{2AA_{18}m_1-4S_0A_{54}m_1^{\ 2}}{\left(\frac{4}{S_c}\right)m_1^{\ 2}-2m_1-(n+\gamma)}\quad A_{79}=\frac{2AA_{19}m_3-4S_0A_{55}m_3^{\ 2}}{\left(\frac{4}{S_c}\right)m_3^{\ 2}-2m_3-(n+\gamma)}\quad A_{80}=\frac{AA_{20}m_1-S_0A_{56}m_1^{\ 2}}{\left(\frac{1}{S_c}\right)m_1^{\ 2}-m_1-(n+\gamma)}$$

$$A_{81}=\frac{AA_{21}(m_1+m_3)-S_0A_{57}(m_1+m_3)^2}{\left(\frac{1}{S_c}\right)(m_1+m_3)^2-(m_1+m_3)-(n+\gamma)}\quad A_{82}=\frac{AA_{22}m_3-S_0A_{58}m_3^{\ 2}}{\left(\frac{1}{S_c}\right)m_3^{\ 2}-m_3-(n+\gamma)}$$

$$A_{83}=\frac{AA_{23}m_4-S_0A_{59}m_4^{\ 2}}{\left(\frac{1}{S_c}\right)m_4^{\ 2}-m_4-(n+\gamma)}\quad A_{84}=\frac{AA_{24}(m_1+m_4)-S_0A_{60}(m_1+m_4)^2}{\left(\frac{1}{S_c}\right)(m_1+m_4)^2-(m_1+m_4)-(n+\gamma)}$$

$$A_{85}=\frac{AA_{25}(m_3+m_4)-S_0A_{61}(m_3+m_4)^2}{\left(\frac{1}{S_c}\right)(m_3+m_4)^2-(m_3+m_4)-(n+\gamma)}\quad A_{86}=\frac{-S_0A_{74}m_{10}^{\;2}}{\left(\frac{1}{S_c}\right)m_{10}^{\;2}-m_{10}-(n+\gamma)}$$

$$A_{87}=\frac{-S_0A_{62}(m_8+m_4)^2}{\left(\frac{1}{S_c}\right)(m_8+m_4)^2-(m_8+m_4)-(n+\gamma)}\quad A_{88}=\frac{-S_0A_{63}(m_2+m_4)^2}{\left(\frac{1}{S_c}\right)(m_2+m_4)^2-(m_2+m_4)-(n+\gamma)}$$

$$A_{89}=\frac{-S_0A_{64}(m_7+m_4)^2}{\left(\frac{1}{S_c}\right)(m_7+m_4)^2-(m_7+m_4)-(n+\gamma)}\quad A_{90}=\frac{-S_0A_{65}m_8^{\;2}}{\left(\frac{1}{S_c}\right)m_8^{\;2}-m_8-(n+\gamma)}$$

$$A_{91}=\frac{-S_0A_{66}m_2^{\;2}}{\left(\frac{1}{S_c}\right)m_2^{\;2}-m_2-(n+\gamma)}\quad A_{92}=\frac{-S_0A_{67}m_7^{\;2}}{\left(\frac{1}{S_c}\right)m_7^{\;2}-m_7-(n+\gamma)}$$

$$A_{93}=\frac{-S_0A_{68}(m_8+m_1)^2}{\left(\frac{1}{S_c}\right)(m_8+m_1)^2-(m_8+m_1)-(n+\gamma)}\quad A_{94}=\frac{-S_0A_{69}(m_2+m_1)^2}{\left(\frac{1}{S_c}\right)(m_2+m_1)^2-(m_2+m_1)-(n+\gamma)}$$

$$A_{95}=\frac{-S_0A_{70}(m_7+m_1)^2}{\left(\frac{1}{S_c}\right)(m_7+m_1)^2-(m_7+m_1)-(n+\gamma)}\quad A_{96}=\frac{-S_0A_{71}(m_8+m_3)^2}{\left(\frac{1}{S_c}\right)(m_8+m_3)^2-(m_8+m_3)-(n+\gamma)}$$

$$A_{97}=\frac{-S_0A_{72}(m_2+m_3)^2}{\left(\frac{1}{S_c}\right)(m_2+m_3)^2-(m_2+m_3)-(n+\gamma)}\quad A_{98}=\frac{-S_0A_{73}(m_7+m_3)^2}{\left(\frac{1}{S_c}\right)(m_7+m_3)^2-(m_7+m_3)-(n+\gamma)}$$

$$A_{99}=-A_{75}-A_{76}-A_{77}-A_{78}-A_{79}-A_{80}-A_{81}-A_{82}-A_{83}-A_{84}-A_{85}-A_{86}-A_{87}-A_{88}-A_{89}-A_{90}-A_{91}$$
$$-A_{92}-A_{93}-A_{94}-A_{95}-A_{96}-A_{97}-A_{98}$$

$$A_{100}=\frac{AA_{51}m_9}{m_9^{\;2}-m_9-(n+N)}\quad A_{101}=\frac{AA_{41}m_5-G_TA_{52}-G_cA_{76}}{m_5^{\;2}-m_5-(n+N)}\quad A_{102}=\frac{2AA_{42}m_4-G_TA_{53}-G_cA_{77}}{4m_4^{\;2}-2m_4-(n+N)}$$

$$A_{103}=\frac{2AA_{43}m_1-G_TA_{54}-G_cA_{78}}{4m_1^{\;2}-2m_1-(n+N)}\quad A_{104}=\frac{2AA_{44}m_3-G_TA_{55}-G_cA_{79}}{4m_3^{\;2}-2m_3-(n+N)}$$

$$A_{105}=\frac{AA_{45}m_1-G_TA_{56}-G_cA_{80}}{m_1^{\;2}-m_1-(n+N)}$$

$$A_{106}=\frac{AA_{46}(m_1+m_3)-G_TA_{57}-G_cA_{81}}{(m_1+m_3)^2-(m_1+m_3)-(n+N)}\quad A_{107}=\frac{AA_{47}m_3-G_TA_{58}-G_cA_{82}}{m_3^{\;2}-m_3-(n+N)}$$

$$A_{108}=\frac{AA_{48}m_4-G_TA_{59}-G_cA_{83}}{m_4^{\;2}-m_4-(n+N)}\quad A_{109}=\frac{AA_{49}(m_1+m_4)-G_TA_{60}-G_cA_{84}}{(m_1+m_4)^2-(m_1+m_4)-(n+N)}$$

$$A_{110} = \frac{AA_{50}(m_3+m_4) - G_T A_{61} - G_c A_{85}}{(m_3+m_4)^2 - (m_3+m_4) - (n+N)} \quad A_{111} = \frac{-G_T A_{74} - G_c A_{86}}{m_{10}^{\ 2} - m_{10} - (n+N)}$$

$$A_{112} = \frac{-G_T A_{62} - G_c A_{87}}{(m_4+m_8)^2 - (m_4+m_8) - (n+N)} \quad A_{113} = \frac{-G_T A_{63} - G_c A_{88}}{(m_4+m_2)^2 - (m_4+m_2) - (n+N)}$$

$$A_{114} = \frac{-G_T A_{64} - G_c A_{89}}{(m_4+m_7)^2 - (m_4+m_7) - (n+N)} \quad A_{115} = \frac{-G_T A_{65} - G_c A_{90}}{m_8^{\ 2} - m_8 - (n+N)} \quad A_{116} = \frac{-G_T A_{66} - G_c A_{91}}{m_2^{\ 2} - m_2 - (n+N)}$$

$$A_{117} = \frac{-G_T A_{67} - G_c A_{92}}{m_7^{\ 2} - m_7 - (n+N)} \quad A_{118} = \frac{-G_T A_{68} - G_c A_{93}}{(m_1+m_8)^2 - (m_1+m_8) - (n+N)}$$

$$A_{119} = \frac{-G_T A_{69} - G_c A_{94}}{(m_1+m_2)^2 - (m_1+m_2) - (n+N)} \quad A_{120} = \frac{-G_T A_{70} - G_c A_{95}}{(m_1+m_7)^2 - (m_1+m_7) - (n+N)}$$

$$A_{121} = \frac{-G_T A_{71} - G_c A_{96}}{(m_3+m_8)^2 - (m_3+m_8) - (n+N)} \quad A_{122} = \frac{-G_T A_{72} - G_c A_{97}}{(m_3+m_2)^2 - (m_3+m_2) - (n+N)}$$

$$A_{123} = \frac{-G_T A_{73} - G_c A_{98}}{(m_3+m_7)^2 - (m_3+m_7) - (n+N)} \quad A_{124} = \frac{-G_c A_{99}}{m_{11}^{\ 2} - m_{11} - (n+N)} \quad A_{125} = \frac{-G_c A_{75}}{m_6^{\ 2} - m_6 - (n+N)}$$

$$A_{126} = -A_{100} - A_{101} - A_{102} - A_{103} - A_{104} - A_{105} - A_{106} - A_{107} - A_{108} - A_{109} - A_{110} - A_{111} - A_{112} - A_{113} - A_{114} - A_{115} - A_{116} - A_{117} - A_{118} - A_{119} - A_{120} - A_{121} - A_{122} - A_{123} - A_{124} - A_{125}$$

$$A_{127} = -m_4 A_{33} - m_1 A_3 - m_3 A_4 - E_c \begin{pmatrix} m_9 A_{51} + m_5 A_{41} + 2m_4 A_{42} + 2m_1 A_{43} + 2m_3 A_{44} + m_1 A_{45} \\ +(m_1+m_3)A_{46} + m_3 A_{47} + m_4 A_{48} + (m_1+m_4)A_{49} + (m_3+m_4)A_{50} \end{pmatrix}$$

$$A_{128} = -m_8 A_{39} - m_4 A_{34} e^{-m_4 y} - m_1 A_{35} - m_3 A_{36} - m_2 A_{37} - m_7 A_{38}$$
$$- E_c \begin{pmatrix} m_{12} A_{126} + m_9 A_{100} + m_5 A_{101} + 2m_4 A_{102} + 2m_1 A_{103} + 2m_3 A_{104} + m_1 A_{105} + (m_3+m_1)A_{106} + m_3 A_{107} + m_4 A_{108} + (m_1+m_4)A_{109} \\ +(m_3+m_4)A_{110} + m_{10} A_{111} + (m_4+m_8)A_{112} + (m_2+m_4)A_{113} + (m_7+m_4)A_{114} + m_8 A_{115} + m_2 A_{116} + m_7 A_{117} + (m_1+m_8)A_{118} \\ +(m_1+m_2)A_{119} + (m_1+m_7)A_{120} + (m_3+m_8)A_{121} + (m_2+m_3)A_{122} + (m_3+m_7)A_{123} + m_{11} A_{124} + m_6 A_{125} \end{pmatrix}$$

$$A_{129} = -m_1 - E_c \begin{pmatrix} m_5 A_{15} + 2m_4 A_6 + 2m_1 A_7 + 2m_3 A_8 + m_1 A_9 + (m_1+m_3)A_{10} \\ +m_3 A_{11} + m_4 A_{12} + (m_1+m_4)A_{13} + (m_3+m_4)A_{14} \end{pmatrix}$$

$$A_{130} = -m_2 A_{27} - m_1 A_1$$
$$- E_c \begin{pmatrix} m_{10} A_{74} + m_5 A_{52} + 2m_4 A_{53} + 2m_1 A_{54} + 2m_3 A_{55} + m_1 A_{56} + (m_3+m_1)A_{57} + m_3 A_{58} + m_4 A_{59} + (m_1+m_4)A_{60} \\ +(m_3+m_4)A_{61} + (m_4+m_8)A_{62} + (m_2+m_4)A_{63} + (m_7+m_4)A_{64} + m_8 A_{65} + m_2 A_{66} + m_7 A_{67} \\ +(m_1+m_8)A_{68} + (m_1+m_2)A_{69} + (m_1+m_7)A_{70} + (m_3+m_8)A_{71} + (m_2+m_3)A_{72} + (m_3+m_7)A_{73} \end{pmatrix}$$

$$A_{131} = -m_3 A_{28} - m_1 A_2 - E_c \begin{pmatrix} m_6 A_{26} + m_5 A_{16} + 2m_4 A_{17} + 2m_1 A_{18} + 2m_3 A_{19} + m_1 A_{20} \\ +(m_1+m_3)A_{21} + m_3 A_{22} + m_4 A_{23} + (m_1+m_4)A_{24} + (m_3+m_4)A_{25} \end{pmatrix}$$